Generalized Interference Model*†

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A crossing-symmetric Regge-pole model of the generalized interference type is discussed. The model consists of a sum of infinitely many Regge poles in each channel, corresponding to a leading Regge trajectory and its parallel daughters. All the usual requirements are satisfied by the Regge residues, and nonlinear trajections can be introduced without difficulty. At the expense of a loss of physical interpretation, the double-counting problem can be eliminated by identifying the Schmid loops with the direct-channel resonances.

1. INTRODUCTION

RECENTLY there has been a great deal of interest in constructing crossing-symmetric, Regge-behaved models of scattering amplitudes. The desire to construct such models is easily understood, since they would combine two very fundamental theoretical ideas (crossing symmetry and Regge poles) in a single package. If, in addition, the models satisfy the analyticity properties required by S-matrix theory then, presumably, only unitarity would be missing. In this connection, the use of Regge-behaved amplitudes has the well-known advantage that unitarity is not manifestly contradicted at high energies.

The construction of crossing-symmetric Regge-pole models is of importance even from a purely phenomenological point of view. For example, in pp scattering, t-u crossing is equivalent to the Pauli principle for the two protons.[1] For large momentum transfers, especially for those corresponding to scattering near 90°, the Pauli principle has major effects which should not be neglected. Similarly, in πN scattering, the effects of the t channel ($\pi\pi \rightarrow N\bar{N}$) and the u channel ($\pi N \rightarrow \pi N$) are expected to be of comparable importance for large-angle scattering. In the πN problem, the Bose symmetry of the t-channel 2π state has the consequence that whenever u-channel poles are present, the s-channel poles must also be included. Thus, a meaningful phenomenological Regge model of large-angle πN scattering should embody the crossing properties at the outset.

A very interesting crossing-symmetric Regge pole model has been proposed recently by Veneziano.[2] The Veneziano model incorporates analyticity, crossing symmetry, and Regge behavior in a way that is intimately connected with the recently discovered concept of duality.[3,4] (See, however, Ref. 5.) In this paper we study another type of crossing-symmetric, analytic Regge-pole model in which duality is not an essential ingredient. The model is basically a generalized interference model[6] in which the s, t, and u poles occur additively, in separate terms. The terms containing the s poles do not, for example, contribute to the s-channel asymptotic behavior. The model is generally "nondualistic" and it will generally entail "double counting."[3]

The double-counting aspect of the model is eliminated or, at least, minimized for certain values of the parameters involved in the model (e.g., trajectory intercepts). This is achieved by identifying the "Schmid loops"[7] with the direct-channel resonances, leading to a bootstrap condition for the Regge trajectories. The trajectory intercepts obtained in this way agree well with those obtained from finite-energy sum-rule saturation[8,9] or from experiment.

In Sec. 2 the model is written down for the scattering of equal-mass spinless particles with internal symmetries neglected (the generalization to arbitrary spins, masses, and symmetries is straightforward), and the analyticity and asymptotic properties of the full amplitude are outlined.

In Sec. 3 the l-plane properties of the amplitude are studied. It is found that there is an infinite family of Regge poles, consisting of a leading trajectory and its parallel daughters. The analyticity properties of the Regge residues are found to be precisely those required by general Regge-pole theory.[10]

Section 4 contains a short discussion of the problem of duality and double counting in the model. It is argued that double counting can be eliminated by requiring that the total "observed" resonating partial-wave loop be made up partly by an actual resonance pole and partly by the Schmid loop, as in the model of Alessandrini, Amati, and Squires.[11] Some of the difficulties with the physical interpretation of this procedure are also discussed.

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2. GENERALIZED INTERFERENCE MODEL

In this section the model is written down for the simple case of equal-mass, spinless-particle scattering. The kinematics is indicated in Fig. 1. In the sequel s, t, and u are the usual Mandelstam variables, defined as $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $u=(p_1-p_4)^2$. The metric is chosen so that $p_i^2=m^2$.

The scattering amplitude is

$$A(s,t,u)=A^{(s)}(s,t,u)+A^{(t)}(s,t,u)+A^{(u)}(s,t,u)\,, \quad (2.1\text{a})$$

where

$$A^{(s)}(s,t,u)=f(s)[(4m^2-t)^{\alpha(s)}+(4m^2-u)^{\alpha(s)}], \quad (2.1\text{b})$$

$$A^{(t)}(s,t,u)=f(t)[(4m^2-s)^{\alpha(t)}+(4m^2-u)^{\alpha(t)}], \quad (2.1\text{c})$$

$$A^{(u)}(s,t,u)=f(u)[(4m^2-s)^{\alpha(u)}+(4m^2-t)^{\alpha(u)}]. \quad (2.1\text{d})$$

Obviously, $A(s,t,u)$ is symmetric in s, t, and u. If the functions f and α are real-analytic with right-hand cuts from $4m^2$ to ∞, then A has the cut structure required by S-matrix theory.[12] Later it will be seen that the Regge poles are α, $\alpha-1$, $\alpha-2$, $\cdots$, as might be expected from the form of Eqs. (2.1). In (2.1a), only leading-order terms have been written explicitly. Lower-order terms can be added on in the obvious way.

It is clear that the amplitude A has no poles in s, t, and u unless f has poles. A pole at $s=s_l$ will have a residue which is a polynomial in t if and only if $\alpha(s_l)=l$, a non-negative integer. When this is true, the pole occurs in finitely many partial waves $a_l(s)$, $a_{l-1}(s)$, $\cdots$. Thus, if A is supposed to describe an amplitude with resonances lying on a Regge trajectory $\alpha(s)$, then $f(s)$ must contain a pole factor such as $\Gamma(1-\alpha(s))$. Because of the structure of Eqs. (2.1), the residues of such poles are automatically polynomials in t, even if the trajectory $\alpha(s)$ is nonlinear. This is a feature of the model which could make it well suited to phenomenological applications. The model of Veneziano does not have this property, and it seems to be the case[4] that it must first be "unitarized" before it can treat resonances of finite width.

A necessary property of any scattering amplitude $A(s,t)$ is that of polynomial boundedness, for some range of fixed t, as $|s|\to\infty$. This is needed to guarantee the existence of dispersion relations. For the amplitude A of Eq. (2.1a), this property depends critically on the behavior of the function $f(s)$ for large $|s|$, which is now considered in more detail.

First, it will be assumed throughout that the trajectory function $\alpha(s)$ is "essentially linear" in the sense that

$$\lim_{|s|\to\infty}\frac{\alpha(s)}{as+b}=1 \quad (2.2)$$

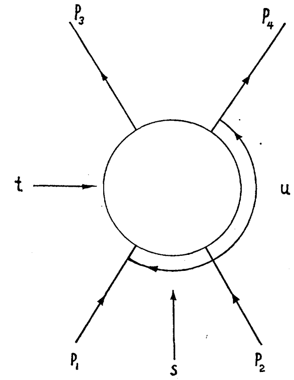

FIG. 1. Kinematics for equal-mass, spinless-particle scattering.

for appropriate a and b, with $a>0$. Then the large-$|s|$ behavior of $A^{(s)}$ is (as $|s|\to\infty$, with t fixed)

$$A^{(s)}(s,t)\to f(s)[(4m^2-t)^{as+b}+e^{at}s^{as+b}]. \quad (2.3)$$

In order that the term $A^{(s)}$ not destroy the polynomial boundedness as $|s|\to\infty$, the function $f(s)$ clearly must decrease faster than s^{-as} in the right-hand wedge defined by $-\frac{1}{2}\pi<\arg s<\frac{1}{2}\pi$. It must also be chosen so that, for a range of t, $f(s)(4m^2-t)^{as}$ is polynomial bounded in the left-hand wedge $\frac{1}{2}\pi<\arg s<\frac{3}{2}\pi$.

An example of a function with the required properties is easily constructed. First, let A and B be real, positive constants. Then the function

$$h(s)=A+\Gamma(Bs) \quad (2.4)$$

is meromorphic in s and has the following asymptotic behavior:

$$h(s)\sim e^{-Bs}(Bs)^{Bs} \quad \text{as} \quad s\to\infty$$

in the right-hand wedge, and

$$h(s)\sim A \quad \text{as} \quad s\to\infty$$

in the left-hand wedges $\frac{1}{2}\pi<\arg s<\pi-\epsilon$ and $\pi+\epsilon<\arg s<\frac{3}{2}\pi$, where $\epsilon>0$ is arbitrarily small. The function $h(s)$ has simple zeros at values s_1, s_2, $\cdots$, with $s_n\to-\infty$ as $n\to\infty$. From the properties of the Γ function near negative integer values of its argument, it follows that

$$s_n=-(n/B)[1+O(1/n!)]. \quad (2.5)$$

Now let

$$h_0(s)=\prod_{n=1}^{\infty}\left(1-\frac{s^2}{\pi^2 s_n^2}\right). \quad (2.6)$$

From (2.5), it follows that the infinite product (2.6) exists and defines $h_0(s)$ as an entire function having simple zeros at $s=s_1$, s_2, $\cdots$. From the infinite-product representation of $\sin\pi Bs$, it follows that

$$h_0(s)=\frac{\sin\pi Bs}{Bs}\prod_{n=1}^{\infty}\left(\frac{1-s^2/\pi^2 s_n^2}{1-B^2s^2/\pi^2n^2}\right). \quad (2.7)$$

Thus, from (2.5) and (2.7) it follows that

$$h_0(s)\xrightarrow[|s|\to\infty]{}\text{const}\,\sin\pi Bs/s. \quad (2.8)$$

[12] That is, the cuts are correctly positioned. The branch points will generally be of the wrong type, however. The right-hand cut is not a cut of the square-root type. This is related to the incorrect threshold properties discussed below.

Therefore,

$$|h_0(s)| \leqslant \text{const } e^{\pi B|s|}/|s| \tag{2.9}$$

as $|s| \to \infty$.

The function

$$\lambda(s) = h_0(s)/h(s) \tag{2.10}$$

is an entire function with the following asymptotic behavior:

$$|\lambda(s)| \sim |(Bs)^{-Bs}| \tag{2.11}$$

as $s \to \infty$ in the right-hand wedge, and

$$|\lambda(s)| \leq \text{const } e^{\pi B|s|}/|s| \tag{2.12}$$

as $s \to \infty$ in the wedges $\frac{1}{2}\pi < \arg s < \pi - \epsilon$ and $\pi + \epsilon < \arg s < \frac{3}{2}\pi$. From the identity $\Gamma(Bs)\Gamma(1-Bs) = \pi/\sin\pi Bs$, it follows that Eq. (2.12) applies also as $s \to \infty$ in the wedge $\pi - \epsilon < \arg s < \pi + \epsilon$.

Finally, let $g(s)$ be a function (having at most a right-hand cut) which carries the resonance poles and which has a mild asymptotic behavior on the first sheet. For example, let g be given by

$$g(s) = \Gamma(1-\alpha(s))/\Gamma_{\text{asymp}}(1-\alpha(s)),$$

which tends to 1 as $|s| \to \infty$ on the first sheet. Then a suitable function $f(s)$ is

$$f(s) = g(s)h_0(s)/h(s). \tag{2.13}$$

With $f(s)$ given by (2.13), and with $B > a$, we have $A^{(s)}(s,t) \to 0$ exponentially in $|s|$ if $t < 4m^2 - e^{\pi B/a}$. The term $A^{(u)}(s,t)$ is easily seen to have this property also; hence, as $|s| \to \infty$, only the term $A^{(t)}(s,t)$ survives. Thus

$$A(s,t) \xrightarrow[|s|\to\infty]{} A^{(t)}(s,t) \simeq f(t)(1+e^{-i\pi\alpha(t)})s^{\alpha(t)}. \tag{2.14}$$

This is just ordinary Regge-asymptotic behavior, so the existence of dispersion relations is assured. By choosing $f(s)$ to decrease sufficiently fast as s increases, the asymptotic behavior (2.14) can be made to apply at quite low values of s—for example, after the first few resonances.

The specific function $f(s)$ of Eq. (2.13) is intended to be only an *example* of a function of the required type. The restriction to the range $t < 4m^2 - e^{\pi B/a}$ could be relaxed by constructing a different function $f(s)$. Also, the function (2.13) is unnecessarily complicated for practical applications. For numerical purposes it would be much simpler to use a function which behaves properly along the real axis, but which might have unwanted singularities out in the complex plane. For example, the function

$$f(s) = (4m^2-s)^{-ks}/[1+(4m^2-s)^{-ks}] \tag{2.15}$$

has the asymptotic behavior

$$\begin{aligned} f(s) &\to (-s)^{-ks} \quad \text{as} \quad s \to +\infty \\ &\to \quad 1 \quad\quad\;\; \text{as} \quad s \to -\infty, \end{aligned} \tag{2.16}$$

but has extra singularities arising from a vanishing denominator for complex values of s.

By the same considerations which led to Eq. (2.14), it follows that the high-energy behavior in the backward direction results from the term $A^{(u)}$ (as $s \to \infty$, with u fixed):

$$A(s,t) \to A^{(u)}(s,t) \simeq f(u)(1+e^{-i\pi\alpha(u)})s^{\alpha(u)}. \tag{2.17}$$

Another limit of physical interest is the high-energy, fixed-angle limit. Here, s, t, and u all become large together, and the ordinary Regge expansion might not hold. The limit in this case must be extracted with more care. In particular, it is convenient to establish conditions under which $A^{(t)}$ and $A^{(u)}$ dominate in this limit. Using (2.2), we have

$$A^{(s)}(s,t) \xrightarrow[z\text{ fixed}]{} f(s)(\tfrac{1}{2}s)^{\alpha(s)} \times[(1-z)^{\alpha(s)}+(1+z)^{\alpha(s)}] \tag{2.18}$$

and

$$A^{(t)}(s,t) \to f(-\tfrac{1}{2}s(1-z)) \times\{(-s)^{b-\frac{1}{2}as(1-z)}+[\tfrac{1}{2}s(z+1)]^{b-\frac{1}{2}as(1-z)}\}, \tag{2.19}$$

where $z = \cos\theta_s$ is the cosine of the scattering angle in the center-of-mass frame. For the special case $\theta_s = 90°$ ($z=0$), it follows that

$$\left(\frac{A^{(s)}(s,t)}{A^{(t)}(s,t)}\right)_{\theta=90°} \simeq \frac{f(s)}{f(-\frac{1}{2}s)}(\tfrac{1}{2}s)^{\frac{3}{2}as}\times(\text{phase factor}). \tag{2.20}$$

If the f of Eq. (2.15) is used, the dominance of $A^{(t)}$ over $A^{(s)}$ is guaranteed if $k > \frac{3}{2}a$. In this case, $A^{(u)}$ also dominates $A^{(s)}$, and the fixed angle, high-energy behavior becomes

$$\begin{aligned} A(s,t) \to\; & f(\tfrac{1}{2}s(z-1))\{(4m^2-s)^{b+\frac{1}{2}as(z-1)} \\ &+[\tfrac{1}{2}s(1+z)]^{b+\frac{1}{2}as(z-1)}\}+f(-\tfrac{1}{2}s(z+1)) \\ &\times\{(4m^2-s)^{b-\frac{1}{2}as(z+1)}+[\tfrac{1}{2}s(1-z)]^{b-\frac{1}{2}as(z+1)}\}. \end{aligned} \tag{2.21}$$

The behavior indicated in Eq. (2.21) is generally quite different from the fixed-z behavior in a model of the Huang-Pinsky[1] type. In the latter model, the high-energy amplitude is written as

$$A(s,t) \simeq f(t)(1+e^{-i\pi\alpha(t)})s^{\alpha(t)} + f(u)(1+e^{-i\pi\alpha(u)})s^{\alpha(u)}. \tag{2.22}$$

Then $t \simeq \frac{1}{2}s(z-1)$ and $u \simeq -\frac{1}{2}s(z+1)$ are substituted directly into (2.22). The difference between (2.21) and (2.22) resides in the effects of the infinitely many daughter trajectories.

It is clear that $A(s,t)$ has incorrect threshold properties, since $(4m^2-s)^{\alpha(t)}$ becomes infinite as $s \to 4m^2$ for $\alpha(t) < 0$. In the physical region, however, the threshold behavior is finite if $\alpha(0) > 0$. Unfortunately, there is no choice of $\alpha(0)$ which will allow $A(s,t)$ to behave like $(s-4m^2)^{1/2}$ near threshold, even in the physical region. This is a serious flaw of the model (2.1).

3. l-PLANE PROPERTIES

In this section the l-plane properties of the amplitude (2.1) are studied. In fact, consideration is restricted to the term $A^{(s)}$, the terms $A^{(t)}$ and $A^{(u)}$ being regarded as part of the l-plane "background."

The term $A^{(s)}(s,t)$ can be written as

$$A^{(s)}(s,t)=\phi(s)\{[\eta(s)-z]^{\alpha(s)}+[\eta(s)+z]^{\alpha(s)}\}, \quad (3.1)$$

where

$$\phi(s)=f(s)[\tfrac{1}{2}(s-4m^2)]^{\alpha(s)} \quad (3.2)$$

and

$$\eta(s)=(s+4m^2)/(s-4m^2). \quad (3.3)$$

The Regge-pole properties are discussed by calculating explicitly the Froissart-Gribov continuation of $A^{(s)}$. We begin in a region of s such that $\mathrm{Re}\alpha(s)<0$, then analytically continue the final answer. In this s region, $A^{(s)}$ obeys the following dispersion relation:

$$A^{(s)}=-\frac{\phi(s)}{\pi}\sin\pi\alpha(s)\int_{\eta(s)}^{\infty}(z'-\eta)^{\alpha}\times\left(\frac{1}{z'-z}+\frac{1}{z'+z}\right)dz'. \quad (3.4)$$

The Froissart-Gribov continuation is now easily found:

$$a_l^-(s)\equiv 0 \quad (3.5)$$

and

$$a_l^+(s)=-(2/\pi)\phi(s)\sin\pi\alpha(s)I, \quad (3.6)$$

where

$$I=\int_{\eta(s)}^{\infty}[z-\eta(s)]^{\alpha(s)}Q_l(z)dz. \quad (3.7)$$

Here, a_l^+ and a_l^- are, respectively, the positive- and negative-signature partial-wave amplitudes. In Eq. (3.7), Q_l is a Legendre function of the second kind.

The integral (3.7) has been evaluated by Martin,[13] who obtains

$$I=\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}l-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}l+\frac{1}{2}-\frac{1}{2}\alpha)}{2^{\alpha+2}\Gamma(l+\frac{3}{2})\eta^{l-\alpha}}\times F(\tfrac{1}{2}l-\tfrac{1}{2}\alpha,\tfrac{1}{2}l+\tfrac{1}{2}-\tfrac{1}{2}\alpha;l+\tfrac{3}{2};1/\eta^2), \quad (3.8)$$

where $F(\cdots)$ denotes the hypergeometric function. The result (3.6), together with (3.8), now holds for any l and s. It is seen immediately that $a_l^+(s)$ has simple l poles at $l=\alpha, \alpha-2, \alpha-4, \cdots$, from the factor $\Gamma(\frac{1}{2}l-\frac{1}{2}\alpha)$, and at $l=\alpha-1, \alpha-3, \alpha-5, \cdots$, from the factor $\Gamma(\frac{1}{2}l+\frac{1}{2}-\frac{1}{2}\alpha)$. Since these poles all occur in the amplitude $a_l^+(s)$, they are all positive-signature Regge poles. In general Regge theory,[10] the odd Freedman-Wang[14] daughters and the even daughters have opposite signature. The reason that there is no contradiction with the above results is that we are dealing with the equal-mass case, for which the odd daughters are unnecessary. The amplitude (3.1) exhibits *more* daughters than is required by the general theory, but the general theory is not actually *contradicted*.

In the general-mass case, when an amplitude similar to (2.1) is written down and then partial-wave analyzed, it is found that odd and even daughters of *both* signatures occur. Here again, there is no contradiction with the general Freedman-Wang results.

From Eqs. (3.6) and (3.8) it follows that the Regge poles at $l=\alpha, \alpha-1, \cdots$, are the *only* singularities of $A^{(s)}$ in the finite l plane. In particular, $A^{(s)}$ exhibits no fixed poles at the nonsense wrong-signature points, because $A^{(s)}$ itself has no third double-spectral function ρ^{tu}.[15] On the other hand, it is expected that the term $A^{(t)}+A^{(u)}$ will, upon being partial-wave-analyzed, exhibit such fixed poles because this term does have a nonvanishing ρ^{tu}.[15] However, because the Regge poles and the fixed poles occur additively, the fixed poles are not reflected in the Regge residues. Thus, the Regge dips associated with the signature factor $1+e^{-i\pi\alpha(s)}$ will be fully operative as $t\to\infty$ with fixed s [see also Eq. (2 14)]. This is an explicit example of a recent general result of Oehme[16] which shows that fixed poles do not necessitate the existence of the Mandelstam-Wang[17] poles in the Regge residues.

The Froissart-Gribov continuation (3.6) also shows explicitly that the Mandelstam symmetry[18]

$$a_l^+(s)=a_{-l-1}^+(s), \quad l=\text{half-integer} \quad (3.9)$$

is satisfied for $A^{(s)}$. This means that $A^{(s)}$ "Reggeizes" by means of Mandelstam's version[18] of the Sommerfeld-Watson transform, leading to an expression for $A^{(s)}$ of the form

$$A^{(s)}=\sum_{n=0}^{\infty}c_n(s)[1+(-1)^n e^{-i\pi\alpha(s)}]Q_{-\alpha(s)-1+n}(z). \quad (3.10)$$

This is also apparent from the recent work of Khuri,[19] because the functional form of Eq. (3.1) is essentially the same as that of the Khuri model.

The Regge residues $\beta_n(s)$ of the Regge poles at $l=\alpha_n(s)\equiv\alpha(s)-n$ are easily found:

$$\beta_{2n}(s)=\frac{(-1)^{n+1}\sin\pi\alpha}{\pi 2^{\alpha}}\phi\eta^{2n}\frac{\Gamma(\frac{1}{2}-n)\Gamma(\alpha+1)}{n!\Gamma(\alpha_{2n}+\frac{3}{2})}\times F\left(-n,\tfrac{1}{2}-n;\alpha_{2n}+\tfrac{3}{2};\frac{1}{\eta^2}\right), \quad (3.11)$$

with similar results for the odd residues β_{2n+1}. The hypergeometric function in Eq. (3.11) is just a polynomial of degree n in $(1/\eta^2)$. The leading residue is

$$\beta_0(s)=-\frac{\phi(s)\sin\pi\alpha(s)}{\pi 2^{\alpha(s)}}\frac{\Gamma(\frac{1}{2})}{\Gamma(\alpha(s)+\frac{3}{2})}. \quad (3.12)$$

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It follows from (3.12) and (3.2) that the function

$$\gamma_0(s)=\beta_0(s)/(s-4m^2)^{\alpha(s)} \tag{3.13}$$

is real-analytic, and finite at $s=4m^2$, as required by general Regge theory.[10] Using (3.3), it is not difficult to show that the same result holds also for the nth residue:

$$\gamma_n(s)=\beta_n(s)/(s-4m^2)^{\alpha_n(s)} \tag{3.14}$$

is real-analytic, and finite at $s=4m^2$. Furthermore, it is apparent that the residues $\beta_n(s)$ contain the factors $1/\Gamma(\alpha_n+\frac{3}{2})$ which are required[10] by the Mandelstam symmetry condition (3.9).

Equation (3.6) shows immediately that all the residues β_n contain the factor $\sin\pi\alpha(s)$. By itself, this would tend to make all the residues vanish when $\alpha(s)$ passes through integer values. In order to obtain true *resonances* on the Regge trajectories, the residues must be finite at non-negative integer values of α. This requires that $\phi(s)$ contain explicit pole factors which cancel these zeros. If such pole factors are simply deleted, it would not be unnatural to have true Regge poles without having any resonances. In such a case, the direct-channel Regge terms would actually correspond to a "nonresonating background." This is a somewhat peculiar situation, but it does not seem to be in contradiction with any general principles. In fact, in phenomenological applications of Regge theory, zeros are often inserted by hand in residue functions at such values as $\alpha=0$ or 1 (in order to eliminate ghost states). The only difference in the present case is that the residue would vanish at *all* non-negative integer values of α, instead of at just a few such values.

4. DUALITY AND DOUBLE COUNTING

In order to understand how double counting can be eliminated in the model, it is convenient to review very briefly the concept of duality.[3–5] By duality, we mean that the "nondiffractive part" of an amplitude can be represented by means of resonances alone.[20,21] This is *strong duality* in the nomenclature of Ref. 5. If, in addition, we adopt the viewpoint that resonances are identified with loops in the Argand diagrams of partial-wave amplitudes,[22] then the statement of duality becomes the following: Except for diffraction, an amplitude can be represented completely by means of loops in partial-wave amplitudes. In connection with this statement, it must be understood that the *only* partial-wave loops required are those which correspond to the usual resonances lying on the direct-channel Regge trajectories (including daughters).

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The generalized interference model commits double counting because it gives rise to extra partial-wave loops which are not accounted for by the direct-channel poles. These are the famous Schmid loops,[7] arising from crossed-channel poles. To eliminate double counting, we must impose the condition that the two different sets of loops coalesce, i.e., that the positions of the Schmid loops and the s-channel loops coincide. This condition is apparently quite well satisfied in reality,[7,23] and it seems to be satisfied also by theoretical models based on finite-energy sum rules.[9] It is, of course, a *bootstrap* condition, because it is satisfied only for special values of the trajectory intercepts (for a given slope). Once this condition is satisfied, the double counting is eliminated as in Ref. 11: The residue of the s pole is determined by taking the difference between the "full" resonance loop and the Schmid loop. Unfortunately, the full resonance loop must be obtained somehow from outside the model, so that a complete bootstrap calculation is not possible within the confines of the model itself.

The procedure outlined above has the disadvantage that the physical interpretation of the partial-wave loops is completely lost. In fact, it actually misses the point of the duality concept, which is that an amplitude is describable completely in terms of the formation and decay of unstable particles (resonances). The partial-wave loops are supposed to be more than just loops; they are supposed to reflect the formation of long-lived virtual states. The residues of the poles are supposed to describe the coupling of the external particles to the virtual, unstable particles, and these same residues can occur in many different physical processes (except for Clebsch-Gordan coefficients). None of these suppositions holds for the generalized interference model. Here, the loops do not completely reflect the formation of unstable particles. The residues of the s poles would not be the same as those determined from the isobar model and hence would differ greatly in different processes.[23] The residues might even be negative in some cases. The conclusion to be drawn from these remarks is that the generalized interference model is not well suited to the incorporation of duality, because the double counting is eliminated only at the expense of a complete loss of physical interpretation.

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[23] C. Schmid, Nuovo Cimento **61A**, 289 (1969).